Measurement of Fundamental Constants using Johnson and Shot Noise

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In this paper, we examine the relationship between noise in an electric circuit and three fundamental constants of nature. Using the Nyquist formula, we find a relation between the Johnson noise and the resistance in the circuit. This leads to an estimate of Boltzmann's constant k_B. The measurement of Johnson noise as a function of temperature gives another estimate of k_B, and also a measurement of the Celsius temperature of absolute zero. Finally, the measurement of shot noise from a photodiode gives us a value for the charge of an electron e. Our results all gave the correct order of magnitude for these constants, however, we were often several standard deviations off of the accepted values. This lead us to conclude that there was some amount of systematic error in our system that was unaccounted for in our measurements, and we discuss possible sources for this error.

1. INTRODUCTION

Thermodynamics serves as a link connecting microscopic properties of physical objects and the macroscopic behavior of an ensemble of these objects. When the predictions of thermodynamics are applied to the physics of conductors and circuits, we find a series of relationships between the voltages observed in the circuit and several fundamental constants. These relationships arise due to two types of noise caused by thermal fluctuations in the circuitry. The first type, Johnson noise, is present in any circuit containing a finite resistance. The second type, Shot noise, arises from discrete passages of charge carriers, such as during the emissions of a photoelectron. In the following experiments, we will describe techniques for measuring these two types of noise, and discuss how they relate to three fundamental constants: Boltzmann's constant k_B, the Celsius temperature of absolute zero, and the charge of the electron e.

2. JOHNSON NOISE

2.1. Theory

J. B. Johnson performed the first measurement of the noise that arises from thermal fluctuations within a resistor [1], and using the values obtained in these measurements, Harry Nyquist provided a theoretical description of Johnson noise. The formula obtained in Nyquist's paper [2] for the mean squared voltage arising due to thermally induced currents in the frequency range f_j and $f_j + df$ is

$$dV_j^2 = 4Rk_BTdf \tag{1}$$

Nyquist arrived at this result as follows. The second law of thermodynamics states that there can be no net transfer of heat between two systems in thermal equilibrium. Using this result, we can conclude that the power transferred between two connected resistors due to thermally induced currents must net to zero at any frequency, regardless of the microscopic properties of the resistor. This leads to the results that the voltage due to thermal excitations can depend only on the system's resistance, temperature, and frequency of the current.

Next, we note that the energy of each mode of vibration has energy contained in the electromagnetic field in, for example, the transmission line connecting the two resistors. The Hamiltonian for the system will then contain a term depending on the magnitude of the electric and magnetic fields: $(E^2 + B^2)/8\pi$. We note that since $hf \ll k_BT$ for our experiment, we need not consider a quantum mechanical description of the thermodynamics of the system. The equipartition theorem of thermodynamics states that each term of the Hamiltonian that depends only on the square of a canonical position or momentum variable (i.e. E and B), contributes $k_BT/2$ to the total internal energy of the system, allowing each mode to contribute k_BT to the internal energy. By calculating the power transferred to the transmission line using this expression for energy, we arrive at the result given in Equation 1.

More generally, for a circuit with a resistor of resistance R and a shunting capacitance C, circuit theory allows us to conclude that the Nyquist formula becomes

$$dV^2 = 4R_fk_BTdf \tag{2}$$

with

$$R_f = \frac{R}{1 + (2\pi fCR)^2} \tag{3}$$

2.2. Experimental Apparatus

A schematic of the experimental setup for Johnson noise measurement is shown in Figure 1. A resistor is attached to two alligator clips that protrude from the Johnson noise box, and a metal beaker is used to shield

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the resistor from external electromotive forces during the measurement. A switch connecting the circuit to an ohmmeter allows for a measurement of the resistance. During the Johnson noise measurement, this connection is switched off and the ohmmeter is disconnected to reduce external noise. Another switch shorts the circuit to the resistor, which allows us to distinguish between the measurement of the Johnson noise of our resistor and the intrinsic Johnson noise of the setup. The signal from the resistor is fed into an SR560 low noise preamp which serves as an amplifier of gain 500 and a high-pass filter with cutoff frequency of 1 kHz. The signal then enters a Krohn-Hite Model 3988 programmable filter, which serves as a low pass filter with cutoff at 50 kHz. The signal then enters the oscilloscope where we can measure the $V_{\rm rms}$.

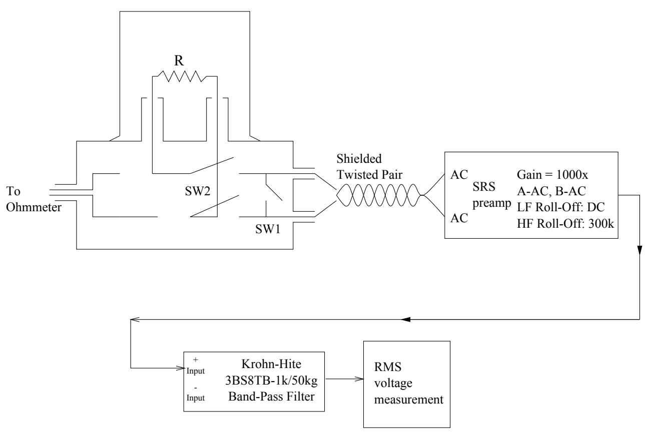

FIG. 1: Diagram of the apparatus for measuring Johnson noise. Image from [3].

In order to test the predictions of Nyquist's theory of Johnson noise, we need to know the gain as a function of frequency of our band-pass filter, as well as the shunting capacitance seen by the resistor. The gain curve is measured using a function generator to send signals of different frequencies through the measurement chain. The ratio of the input and output $V_{\rm rms}$ gives the gain at that frequency. Figure 2 shows the gain squared as a function of frequency. The measurement of the shunting capacitance is straightforward, and we found a value of $C = 22.7 \pm 1$ pF.

When taking measurements, the $V_{\rm rms}$ measured will be related to the $V_{\rm rms}$ produced by Johnson noise in the resistor and the gain $g(f)$ by

$$dV_{\rm meas}^2 = [g(f)]^2 dV^2 \tag{4}$$

The orthogonality of each Fourier component of the signal allows us to integrate Equation 2 over the range of our band-pass to obtain an expression for the total mean

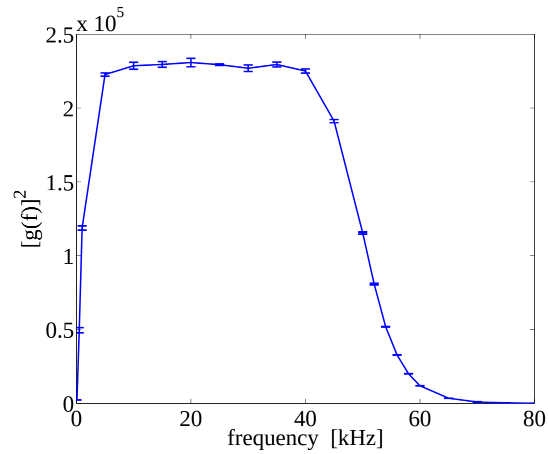

FIG. 2: Gain squared of the Johnson noise measurement chain, with steep dropoffs at 1 kHz and 50 kHz.

squared voltage,

$$V^2 = 4Rk_BTG \tag{5}$$

where

$$G = \int_0^\infty \frac{[g(f)]^2}{1+(2\pi fCR)^2}\,df \tag{6}$$

2.3. Measurements

Our investigation of Johnson noise consists of measuring the mean squared voltage for a variety of resistances, which will yield an estimate of Boltzmann's constant k_B. We also measure the dependence of the mean squared voltage as a function of temperature, which not only provides another estimate of k_B, but also gives the Celsius temperature of absolute zero.

2.3.1. Vary resistance

We took measurements of the Johnson noise for 9 different resistors at room temperature $T = 293$ K. To isolate the noise produced by the resistor, we alternated between measurements of with the resistor connected to the circuit, and the resistor shorted. The mean squared voltage due to the resistor is then given by $V_r^2 - V_s^2$. Five measurements were taken for each resistor to obtain an estimation of the random error of the measurement.

In order to calculate G for a given resistance, we performed a numerical integration of Equation 6 from 0.1 to 80 kHz. We used the trapezoidal method to calculate the integral, which, as shown by Bevington and Robinson [4], has a numerical error associated with each interval of $\frac{df}{12}h''(\xi)$, where h is the integrand, and ξ lies in the interval f and $f+df$. Using finite difference approximations of the second derivative, we found the numerical error to be 7 orders of magnitude less than the random errors of the measurement, so for our purposes we were able to treat the numerical integral as exact.

Our measurements allow us to solve Equation 5 for k_B for each resistor, and we obtained several values for Boltzmann's constant, each with an associated error. Taking a weighted mean that minimizes the residuals over all the resistances, we found a value of $k_B = 1.21 \pm 0.09 \times 10^{-23}\ \mathrm{m^2\,kg\,s^{-2}\,T^{-1}}$. This is within 2σ of the accepted value of 1.38×10^{-23}.

2.3.2. Vary temperature

We also measured the dependence on temperature of the Johnson noise of a single resistor with resistance 817 kΩ. A low temperature measurement at T$= -196°$ C was made by submerging the resistor in a bath of liquid nitrogen. Other measurements were made by putting the resistor in a cylindrical oven, and monitoring the temperature using a mercury glass thermometer. Ten measurements were made at each temperature, again switching between the shorted circuit and the resistor. Calculating G as before, we made a plot of temperature versus $V^2/4RG$ (Figure 3). A linear fit to this data gives another estimate of k_B, as well as an estimate of the Celsius temperature of absolute zero, T_0. We found values of $k_B = 2.90 \pm 0.12 \times 10^{-23}$ and $T_0 = -245° \pm 12°$ C.

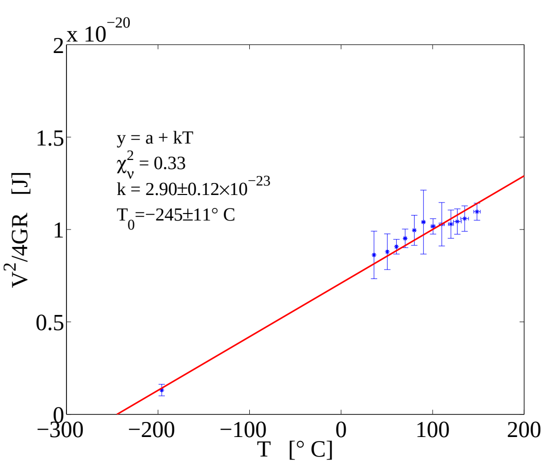

FIG. 3: Measurement of Johnson noise as a function of temperature. The linear fit shown in red has a slope of k_B and an x-intercept of T_0.

Our measurement of absolute zero was 3σ higher than the accepted value of -273, and the measurement of k_B is several standard deviations off. This indicates that there is likely some systematic error that we are not accounting for. One explanation for this error is external noise affecting our measurements, since this experiment is very sensitive to the presence of an external electromagnetic field.

3. SHOT NOISE

3.1. Theory

The noise due to the passage of a discrete charge carrier through the circuit is called shot noise. One example of where it might arise is the emission of photoelectrons in a circuit. The response of the circuit to a single photoelectron is to create an initial spike in the current which quickly settles back to the average current. Several of these events combined create noise in the circuitry which is related to the charge e the particle. Appendix C of [3] provides a derivation of the formula of the current created by shot noise, which is

$$d\langle I^2\rangle = 2eI_{\mathrm{ave}}df \tag{7}$$

Thus we see that the shot noise depends linearly on the charge e of the particle. Thus, a measurement of the shot noise of a system will allow us to calculate the charge of an electron.

3.2. Experimental Apparatus

The apparatus for measuring shot noise is shown in Figure 4. Within the photodiode box, a light bulb of adjustable intensity shines on the photodiode, which creates a current of photoelectrons. The noise measured in this current will be dominated by shot noise. The average DC current is measured using a multimeter, and the signal from the shot noise is amplified within the photodiode box. It then passes through the same preamplifier and band-pass chain as in the Johnson noise measurement, with a gain set at 2000. The AC mean squared voltage is then measured at the oscilloscope.

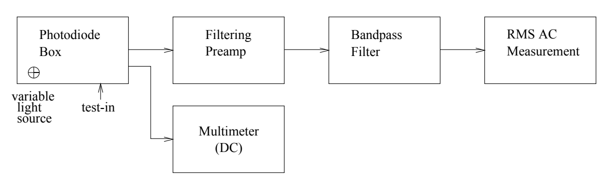

FIG. 4: Block diagram for the shot noise apparatus.

As with the Johnson noise measurement, we calibrated the measurement chain by finding the gain as a function of frequency of our measurement chain. Once again a function generator was used to input a test signal a known frequency into the photodiode box in order to compute the gain $g(f)$. The AC mean squared voltage will then be given by

$$V_0^2 = 2eI_{\mathrm{ave}}R_f^2 \int_0^{\infty} [g(f)]^2\, df + V_A^2 \tag{8}$$

where $R_f = 475$ kΩ, and V_A^2 represents other contributions to noise in the system.

3.3. Measurement

By adjusting the current fed to the light bulb, we can change the intensity of incident light on the photodiode, and thus change the average DC current in the system. For 12 different values of I_{ave} we measured the root mean squared voltage of the AC current coming from the photodiode box. 10 measurements at each value of I_{ave} were taken in order to asses the random errors of the system.

Figure 5 shows a plot of the measured mean squared voltages. The x-axis gives the quantity $2R_f^2 G I_{\text{ave}}$, so that the slope of a line fitted to the data will be e, the charge of an electron. Here, G is the integral of the gain squared over the range of the band-pass filter.

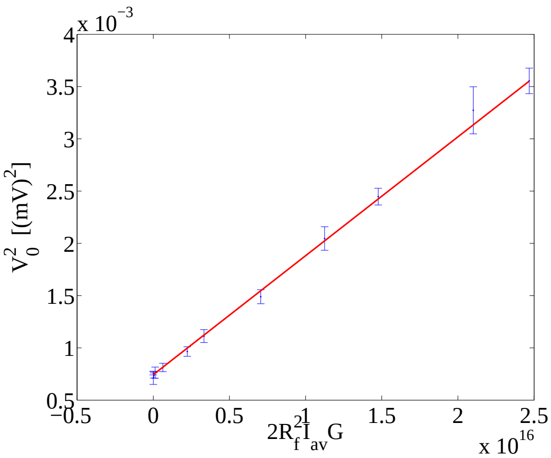

FIG. 5: Plot of shot noise from a photodiode. The linear fit shown in red has a slope of e.

The data yielded a value of $e = 1.138 \pm 0.032 \times 10^{-19}$ Coulombs, and the fit had a reduced chi squared of $\chi^2_\nu = 0.3$. This value for e differs from the accepted value of 1.602×10^{-19} by 30%. As with the Johnson noise measurement, it is likely external noise is interfering with the measurement chain to introduce a systematic error that is skewing our data.

4. CONCLUSIONS

In this experiment, we measured the two types of noise that are inherent in any circuit. We examined the dependence of Johnson noise on resistance, and using our measurements were able to measure Boltzmann's constant to be $k_B = 1.21 \pm 0.09 \times 10^{-23}$ m^2 kg s^{-2} T^{-1}, which is within two standard deviations of the accepted value. Our measurement of Johnson noise as function of temperature yielded gave $k_B = 2.90 \pm 0.12 \times 10^{-23}$ and the Celsius temperature of absolute zero $T_0 = -245° \pm 12°$ C, compared to the accepted value of $-273°$ C. Finally, our measurement of shot noise gave a measurement of the charge of an electron to be $e = 1.138 \pm 0.032 \times 10^{-19}$ Coulombs, which is 30% off the accepted value of 1.602×10^{-19}.

The fact that all our measurements were of the correct order of magnitude, but several standard deviations off the accepted values indicates that there was a systematic error present that we were not accounting for. This error likely arises from external electromagnetic fields, since our apparatus is surrounded by computers and other electronics. Also, the measurements made in this experiment are very sensitive to the configuration of the setup, and change the position of cables can have an effect on the measured noise. These effects combined may have caused our data to have been slightly skewed.

[1] J. B. Johnson, Physical Review **32** (1928).
[2] H. Nyquist, Physical Review **32** (1928).
[3] J. Lab Staff, *Johnson Noise and Shot Noise*, MIT Department of Physics (2010), lab guide.
[4] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, 2003), 3rd ed.

Acknowledgments

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Johnson Noise and Shot Noise Determinations of k_B, T_0, and e

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We examine voltage noise arising from two stochastic processes in electrical systems. We verify Nyquist's theorem of voltage noise by measuring the amplified Johnson noise across a series of coaxial-lead metal film resistors at a variety of temperatures, yielding Boltzmann's constant $k_B = (1.38 \pm 0.06) \times 10^{-23} \mathrm{J} \cdot \mathrm{K}^{-1}$ and absolute zero $T_0 = (-263.49 \pm 35.0 \pm 20.2)^{\circ}$C. By characterizing the shot-nose across a photodiode, we find the electron charge $e = (1.59 \pm 0.08) \times 10^{-19}$C.

1. INTRODUCTION

Unwanted electrical fluctuations—noise—found in signals interferes with measurement accuracy and precision. Noise sources such as stray radio-frequency can be controlled, while others arise from physically-inherent stochastic fluctuations. Johnson noise quantifies the minimum mean-square voltage noise measurable from a signal source with a resistive impedance. Electric charge quantization and its resulting current fluctuations gives rise to shot noise[1].

We describe Johnson and shot noise as stochastic fluctuations in a resistor. To confirm Nyquist's Theorem, we measured an amplified passband of the Johnson noise power spectrum using a variety of resistors, from which Boltzmann's constant k_B and absolute zero T_0 are determined. The Butterworth frequency response of the amplification stage is characterized. We will also determine electron charge e from the shot noise current fluctuations through a photodiode[2].

2. THEORY

We investigate the noise voltage across a resistive impedance arising from the random thermal motions of its electrons. Noise can be represented by the spectral density $J_+(\omega)$ associated with a stochastically-fluctuating emf $V(t)$[3]:

$$\langle V^2 \rangle = \int_0^\infty d\omega J_+(\omega) . \tag{1}$$

The spectral density is calculated by treating a resistor R_0 as an ideal, one-dimensional transmission line with characteristic impedance equal to R_0, operating at thermal equilibrium with temperature T. From the mean energy of a voltage wave propagating with frequency mode ω in the transmission line, we have

$$J_+(\omega) = \frac{2}{\pi} \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} R(\omega) , \tag{2}$$

where $R(\omega)$ is the effective resistance seen by R_0 when accounting for a shunt capacitance C:

$$R(\omega) = \mathrm{Re}\left(R_0 || Z_C\right) = \frac{R_0}{1 + (\omega R_0 C)^2} . \tag{3}$$

In the high thermal limit with $\hbar\omega \ll k_B T$, the RHS Equation 2 simplifies to $(2/\pi) k_B T R(\omega)$, and we recover Nyquist's Theorem for the thermal noise across an impedance with a resistive part:

$$\langle V^2 \rangle = \frac{2k_B T}{\pi} \int_0^\infty d\omega R(\omega) |Y(\omega)|^2 , \tag{4}$$

where $|Y(\omega)|$ is voltage transfer function accounting for signal gain and bandpass filtering needed to measure micro-volt noise. We are interested in measuring this noise to determine k_B.

2.1. Shot Noise

Independent of thermal fluctuations, another source of stochastic noise follows from the quantization of electric charge. In this experiment's photodiode circuit, photo-excitation of electrons from a cathode creates a superposition of current events, each equal in magnitude to its electric charge e, flowing to an anode.

Assuming that electron arrivals at a rate K are independent of each other and therefore form a Poisson process, the time-average of fluctuating current is given by $\langle I \rangle = Ke$, and the average number of arrivals and variance for a random variable n in time interval τ is given by $K\tau$. From Poisson statistics, the shot noise current is given by

$$\langle I^2 \rangle = \langle \Delta I^2 \rangle = \frac{e \langle I \rangle}{\tau} \tag{5}$$

Applying Ohm's law across a resistor R_0 and relating the average time interval τ to the filter bandpass, we have

$$V_0^2 = \frac{e R_0^2 \langle I \rangle}{\pi} \int_0^\infty d\omega |Y(\omega)|^2 . \tag{6}$$

3. EXPERIMENTAL SETUP AND PROCEDURE

Block diagrams depicting the Johnson and shot noise experiments can be found in Figures 1 and 2. Relevant

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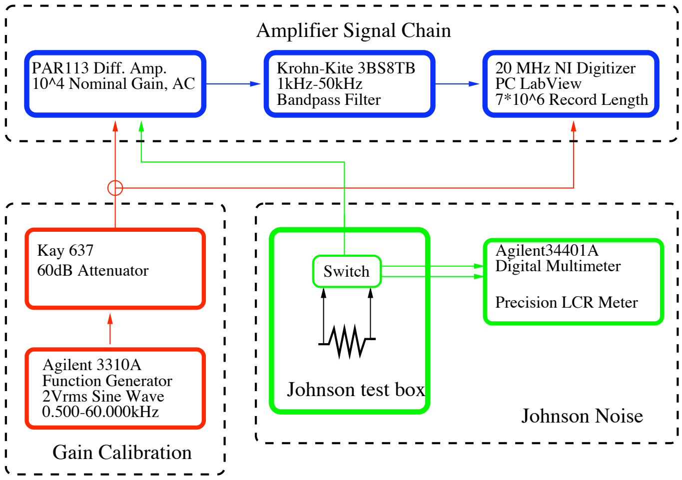

FIG. 1: Johnson noise and calibration apparatus for a fixed temperature and resistance. To increase precision in its signal averaging, the digitizer operates at the maximum 20MHz signal integration and a $7 \cdot 10^6$ sample record length. Household aluminum foil shields twisted, low-impedance coax cabling and the test box. The shield, test box, and amplifier are connected to grounding cable.

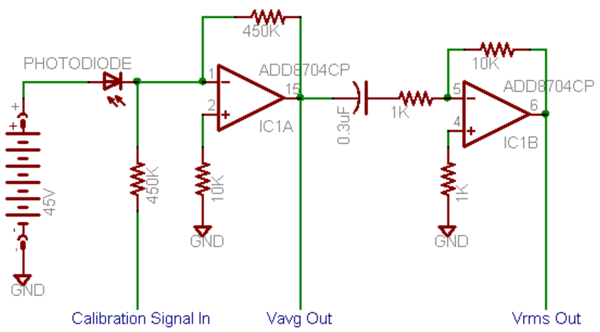

FIG. 2: Shot noise photodiode schematic. Voltage output from this stage is a measure of average current through a photodiode.

parameters applicable to both experiments are discussed in detail with respect to the Johnson noise measurement, followed by a discussion of the shot noise experiment.

3.1. Johnson Noise

We measure the voltage noise and resistances across eight axial-lead, metal film resistors (50kΩ to 800kΩ) mounted in alligator clips on a shielded aluminum test box connected to an amplifier stage. These measurements are repeated at thermal equilibrium with liquid nitrogen (77K), room-temperature ambient air (297.15K), and ambient air (393.15K) heated by a 120V AC Variac-controlled oven. It was not possible to measure the resistor's temperature directly without introducing additional voltage noise and capacitive coupling, therefore temperature was measured in the ambient media of the resistor's immediate vicinity using a toluene-filled glass thermometer and a digital thermometer. Resistance and capacitance measurements at low temperatures were taken quickly to inhibit capacitive charging. Lastly, frequency response for the gain calibration (Section 4.1) is given by the ratio of output and input RMS voltage for the amplifier stage over a 0.500-60.000kHz logarithmic sweep. We determine Boltzmann's constant and an absolute temperature scale with a re-expression of Equation 4,

$$\langle V^2 \rangle = 4 k_B R_0 T G, \tag{7}$$

where G is the effective noise bandwidth for a resistive impedance $R(\omega)$.

Accurately measuring thermal noise require that voltage and current noise in the amplifier, ambient electrical interference, and parasitic and shunt capacitance be minimized and accounted for. To determine amplifier noise, imagine an ideal, noise-free amplifier in series with imaginary voltage and current sources at its input. The amplifier is battery-powered to minimize 60Hz line noise, and source resistances are deliberately chosen to minimize additional noise through the amplifier. Shorting the resistor leads on the test box yields background voltage noise V_S in the amplifier chain. Assuming that sources of voltage noise are statistically uncorrelated, measured Johnson noise is given by $V^2 = V_R^2 - V_S^2$, where V_R^2 is the measured noise across a resistor. Measured background voltage obtained in this manner consistently fell between 11-13mV_{RMS} for all temperatures, while voltage across the resistor ranged between 70-100mV_{RMS} (at 393.15K)[4].

Ambient electrical interference and stray capacitance are minimized through proper grounding, additional shielding, and the use of low impedance cabling. Using the LCR meter on the multimeter ports on the test box, we measure for each resistor the capacitance of coax cabling through the open circuit, and the parasitic capacitance through its junction connection to the alligator clips. Measurements of the capacitance through the open circuit also account for the input impedance (25pF, 100Ω) of the amplifier. Measured capacitance for each resistor ranged from 35-55pF.

3.2. Shot Noise

The shot noise circuit circuit in Figure 2 features a 0.3μF DC–blocking capacitor (high-pass filter at 100Hz) and an op-amp network with a nominal gain of 10. The voltage noise from this network is connected to the same amplifier stage (Figure 1) discussed previously. Frequency gain of the amplification chain was characterized at nominal gains of 1000 and 2000 by disconnecting the photodiode and logarithmically sweeping a function generator through a test-input and measuring the RMS voltage from the second-stage output. As before, the photodiode and pre-amplifier stage were shielded and operated in battery mode to minimize background voltage noise.

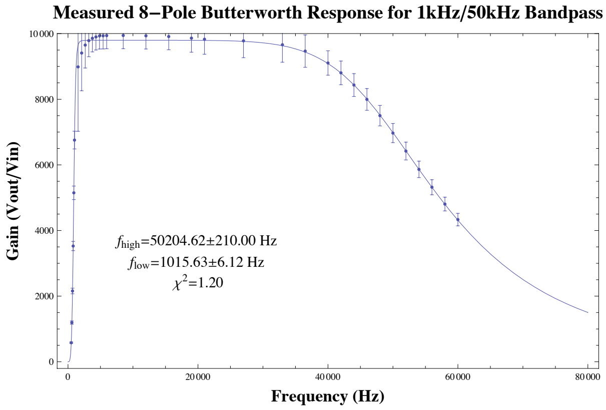

FIG. 3: Gain calibration for 1kHz–50kHz bandpass filter. The discrepancy from theory in the low-pass frequency rise is attributed to manufacturer-specified 0.1dB error.

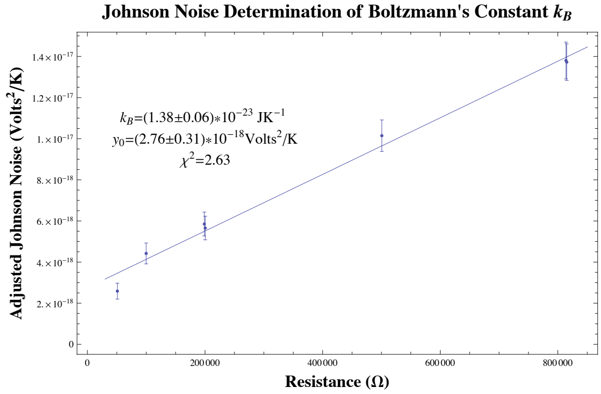

FIG. 4: Fit from adjusted Johnson noise and resistance.

Background noise was subtracted off in quadrature from measured photodiode voltage. For a 450kΩ resistor at a nominal gain of 2000, background noise was approximately 160mV$_{\mathrm{RMS}}$ and extrapolated shot-noise signal ranged between 2-20mV$_{\mathrm{RMS}}$. A fluctuating 1-5mV noise was observed for these measurements, suggesting either a resolution limitation at the digitizer, instability from measuring light bulb current in its transition region, instability in alkaline batteries as a power source, or extraneous noise from the amplifier and ambient electrical interference.

4. RESULTS AND ERROR ANALYSIS

Time variation of measurements, instrument uncertainty, repeatability, and the inherent statistical randomness of our measured processes contribute to systematic and random uncertainty in our error analysis for these measurements. Johnson and shot noise are moments of a random process of the electron. In the context of this process, error is negligible, as the digitizer in the amplification chain ensemble averages over massive sample sets (see caption of Figure 1). Uncertainty in voltage (σ_{V}), capacitance (σ_{C}), and resistance (σ_{R}) measurements were approximately 0.01mV-0.05mV, 0.0005nF, and 0.05kΩ. Error in capacitance and resistance are largely instrumental and also account for time variation at low-temperature measurements. Temperature uncertainty (σ_{T}) was approximately 3.5–5.5K, owing to phase change in ambient media over the hour and the low-precision of the toluene thermometer. Lastly, the manufacturer of our bandpass filter specifies a 0.1dB function deviation (σ_{BP}) from theoretical behavior[5].

4.1. Bandpass Filter Characterization

Johnson and shot noise are dependent on the frequency response and gain of our amplification chain, which we characterize here. The gain in our signal chain for a frequency $f = \omega/2\pi$ and amplifier gain A_0 is described by an 8-pole Butterworth response approximately between two corner frequencies f_{high} and f_{low}:

$$Y(f) = \frac{A_0}{\sqrt{1+\left(\frac{f_{\mathrm{high}}}{f}\right)^8}\sqrt{1+\left(\frac{f}{f_{\mathrm{low}}}\right)^8}} \tag{8}$$

Discounting resistive impedance in Equation 3, a plot of measured gain and its associated 8-pole Butterworth response is in Figure 3. Applying a Marquandt non-linear regression in Mathematica, the amplifier gain is well-characterized by Equation 8 ($\chi^2_\nu = 1.20$), and the corner frequencies are given by $f_{\mathrm{high}} = (50204.6 \pm 210.0)$Hz and $f_{\mathrm{low}} = (1015.63 \pm 6.12)$Hz. Equation 4 implies that equivalent noise bandwidth is determined by integrating over all frequencies. A real bandpass filter actually operates within its specified corner frequencies, therefore values of G used to determine k_B in the following are actually integrated from f_{high} to f_{low}. Integration errors negligible compared to other systematic errors.

Propagated errors the gain calibration are determined for the shot and Johnson measurements as,

$$\sigma^2_Y = 4\left|Y(\omega)\right|^2\left(\frac{\sigma^2_{\mathrm{in}}}{V^2_{\mathrm{in}}} + \frac{\sigma^2_{\mathrm{out}}}{V^2_{\mathrm{out}}}\right) + \sigma^2_{\mathrm{BP}}, \tag{9}$$

$$\sigma^2_G = 4\left(\frac{\left|Y(\omega)\right|^2}{1+(\omega R_0 C)^2}\right)^2\left(\frac{\sigma^2_{Y^2}}{Y^4} + \frac{\sigma^2_R}{R^2} + \frac{\sigma^2_C}{C^2}\right). \tag{10}$$

4.2. Measurement of Boltzmann Constant k_B and Absolute Zero

Figure 4 shows the results of plotting $\langle V^2\rangle/4TG$ against R_0, from whose slope we can determine k_B. Errors from voltage measurements of noise and shorted

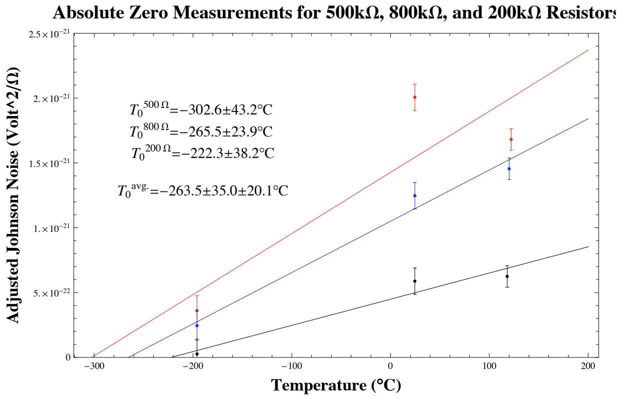

FIG. 5: Fit from adjusted Johnson noise and temperature.

noise add in quadrature to $\sigma^2_{\langle V^2\rangle}$

$$\sigma^2_{\frac{\langle V^2\rangle}{4GT}} = \left(\frac{\langle V^2\rangle}{4GT}\right)^2 \left(\frac{\sigma^2_{\langle V^2\rangle}}{\langle V^2\rangle^2} + \frac{\sigma_T^2}{T^2} + \frac{\sigma_G^2}{G^2}\right). \qquad (11)$$

The experimentally determined value is Boltmann's constant is $k_B = (1.38 \pm 0.06) \times 10^{-23}\mathrm{J}\cdot\mathrm{K}^{-1}$, which agrees excellently with the accepted NIST value of $1.3806504(24) \times 10^{-23}\mathrm{J}\cdot\mathrm{K}^{-1}$ [6]. We expect that our fit line ($\chi^2_\nu = 2.63$) to have a y-intercept at 0 from Equation 7, however our measured y-intercept is given by $(2.76 \pm 0.31) \times 10^{-18}\mathrm{Volts}^2/\mathrm{K}$, which indicates that additional background voltage noise may not be accounted for the quadrature interpolation of Johnson noise, and that additional sources of noise (Johnson and shot noise) may be correlated. Furthermore, note that 50kΩ outlier may be attributed to an impedance mismatch between the source resistor, amplifier, and coax cabling.

Figure 5 shows the results of plotting $\langle V^2\rangle/4RG$ against T for three resistors theoretically valued at 200kΩ, 500kΩ, and 800kΩ, from whose y-intercept we can determine T_0. Error from these points is calculated similarly to Equation 11. Averaging the intercepts for the three resistors, our experimentally determined value of absolute zero is $T_0 = (-263.49 \pm 35.0 \pm 20.2)^\circ\, C$, which agrees within experimental error of the accepted value $T_0 = -273.15K$ [6]. As only three points are used for each of these points, a χ^2 test of confidence is omitted.

Note that a non-equilibrium temperature gradient between a resistor and the ambient environment may result in a departure from the linear temperature dependence of Johnson noise as predicted by Nyquist's theorem. For this reason, systematic error for each of our measurements is larger than it would had it been possible to take measurements thermally coupled to each resistor. As temperature increases, it is more likely for the resistor to approach thermal equilibrium with radiating metal surfaces around it, such as the surface of the oven or the alligator clips, and have an empirical temperature consistently higher than the ambient temperature being measured. Our noise measurements likely overestimated the actual Johnson noise across each resistor for a given temperature.

4.3. Shot Noise Determination of Electron Charge

To determine shot noise, we plot the adjusted shot-noise voltage V/R_fG against photoelectric voltage for a nominal gain of 1000 (χ^2_ν =1.32) and 2000 (χ^2_ν =1.93), determining the electron charge to $e = (1.48 \pm 0.05)\cdot 10^{-19}\mathrm{C}$ and $e = (1.59 \pm 0.08)\cdot 10^{-19}\mathrm{C}$. Repeated measurements of the 1000 gain measurement yielded consistently low values of e, suggesting that fluctuating background noise may be comparable in magnitude to our amplified shot-noise signal. Our measured value with 2000 gain is in excellent agreement with the accepted value of $1.602176487(40)\times 10^{-23}\mathrm{C}$[6].

5. CONCLUSIONS

In summary, we have measured Boltzmann's constant, absolute zero, and the charge of the electron to within experimental error, thereby validating Nyquist's theory of voltage noise from thermal motion of electrons and their quantization. The frequency-response of the amplification chain used in this measurement was a well-characterized Butterworth transfer function.

Our method indicates a number of improvements that future students could make in order to obtain more accurate data. First, measuring shunt and parasitic capacitance should take into account capacitance inherent to the setup as well as that due to the resistor and its junction connection to the test box. Second, measuring the temperature in a thermally-coupled manner without introducing line noise or capacitive coupling is a hurdle that should be addressed. Non-equilibrium behavior of resistors will likely cause actual values of noise to be higher for a measured temperature. Three simple solutions would mitigate these difficulties: first, the use of a conductive platinum resistance thermometer; second, the addition of a thermally stable oil (dibutyl phthalate, paraffin wax, or silicone oil) to a heat bath; and third, to vacuum pump air out of the oven and allow only for thermal transfer between the resistor and oven surface.

Acknowledgments

B. Mookerji thanks C. Herder for his equal contribution to the this experiment and its analysis, and the Junior Lab staff.

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Johnson Noise

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An experiment was performed that determines the Boltzmann constant k and the centigrade temperature of absolute zero by measuring the thermal noise of resistors. The Nyquist theorem provides a quantitative relationship between the thermal electromotive force across a conductor and its resistance and temperature. Measurement of the root-mean-square RMS voltage for a variety of resistors at a fixed temperature was used to calculate the Boltzmann constant. The RMS voltage for a 22.5 kΩ resistor was measured over 300 degree temperature range. This latter data extrapolated to zero centigrade gave an estimate of absolute zero and provided an additional method for determining the Boltzmann constant. The experimentally determined values of the Boltzmann constants, $1.37 \pm 0.06 \times 10^{-23}$ J/K & $1.363 \pm 0.025 \times 10^{-23}$ J/K, and the centigrade temperature of absolute zero, -265.5 ± 6.9°C, are in good agreement with the accepted values.

I. INTRODUCTION

This paper is a full report on the junior lab experiment: **Johnson Noise**. In this experiment, we study the phenomenon of thermal (Johnson) noise as predicted by the Nyquist Theory.

This report has been partitioned into sections accordingly, each discussing a specific aspect of the experiment. Section II discusses the theoretical background relevant to the experiment by deriving the Nyquist Theorem using two different approaches. The experimental apparatus and details of its operation are discussed in section III. Section IV presents the experimental results. Concluding remarks are given in section V.

II. NYQUIST THEORY

Johnson Noise is the mean-square electromotive force in conductors due to thermal agitation of the electromagnetic modes which are coupled to the thermal environment by the charge carriers. The Nyquist Theory is of great importance to experimental physics and in electronics. It gives a quantitative expression for the Johnson Noise generated by a system in thermal equilibrium and is therefore needed in any estimate of the limiting signal-to-noise ratio of an experimental apparatus. In this section, the Nyquist theorem is derived in two ways: first, following the original transmission line derivation, and, second using microscopic arguments [1], [2].

A. Transmission Line Derivation

Consider two conductors each of resistance R at a temperature T connected as depicted in Figure 1. Conductor 1 produces a current I in the circuit equal to the electromotive force due to thermal agitation divided by the total resistance $2R$. This current delivers power to conductor 2 equal to current squared times the resistance. By symmetry, one can deduce that the situation is reciprocal. Conductor 2 produces a similar current which delivers power to conductor 1. Because the two conductors are at the same temperature, the second law of thermodynamics dictates that the power flowing in both directions is equal. I emphasize that no assumption about the nature of conductors has been made.

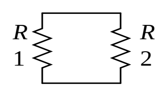

FIG. 1. Two conductors with equal resistance R.

It can be shown that this equilibrium condition holds at any given frequency. Suppose there exists a frequency interval $\Delta\nu_1$ where conductor 1 receives more power than it transmits. We then connect a non-dissipative network with a resonance in the frequency interval $\Delta\nu_1$ between the two conductors (refer to Figure 2). Since the system was in equilibrium prior to inserting the network, it follows that after is insertion more power would be transferred from conductor 2 to conductor 1. However, as the conductors are at the same temperature, this would violate the second law of thermodynamics. The results we have arrived at are important enough to merit summarizing. By eminently reasonable theoretical arguments, we can conclude that the electromotive force due to thermal agitation in conductors are *universal* functions of (refer to Figure 3):

- frequency ν
- resistance R
- temperature T

Experiments performed by Dr. J. B. Johnson in 1928 confirmed the formula which was later derived Dr. H. Nyquist on purely theoretical grounds [3].

The derivation of the mean-square voltage $\langle V^2 \rangle$ across a conductor closely follows Nyquist's original derivation. The problem of determining a quantitative expression for the thermal agitation (i.e. the mean-square voltage) of

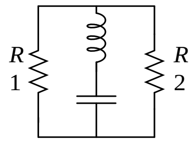

FIG. 2. Two conductors plus resonant circuit.

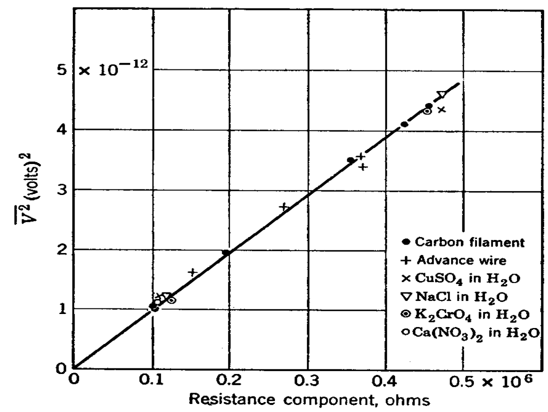

FIG. 3. Voltage-squared vs. resistance component for various types of conductors.

the conductor can be viewed as a simple one-dimensional case of black-body radiation. Consider a lossless one-dimensional transmission line of length L terminated at both ends by conductors with resistance R. The transmission line has been chosen to have a characteristic impedance $Z = R$; consequently any voltage wave propagating along the transmission line is completely absorbed by the terminating resistor without any reflections. Voltage waves of the form $V = V_0 \exp\left[i(k_x x - \omega t)\right]$ propagate down the transmission line at velocity $v = \omega/k_x$. The available number of modes can be calculated by imposing the periodic boundary condition $V(0) = V(L)$ on the propagating voltage waves. The wave vector k_x is related to the length by the relation $k_x L = 2\pi n$ where n is any integer. The density of modes is then,

$$\begin{aligned} D(\omega) &= \frac{1}{L}\frac{dn}{d\omega} \\ &= \frac{1}{L}\frac{dn}{dk_x}\frac{dk_x}{d\omega} \\ &= \frac{1}{2\pi v} \end{aligned} \qquad (1)$$

The mean energy per mode is given by the Planck formula,

$$\langle \varepsilon(\omega) \rangle = \frac{\hbar\omega}{\exp\frac{\hbar\omega}{kT} - 1} \qquad (2)$$

$$\langle \varepsilon(\omega) \rangle \approx kT \qquad (3)$$

where in the last line we made use of the equipartition theorem: in the classical limit, $\hbar\omega \ll kT$, each squared canonical term in the the Hamiltonian contributes $\frac{1}{2}kT$ to the mean energy.[1]

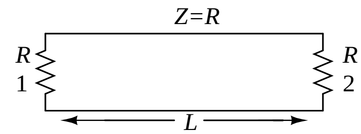

FIG. 4. Lossless transmission line $Z = R$ of length L with matched terminations.

The energy density per unit frequency $U(\omega)$ is then given by the product of the density of modes and the mean energy per mode:[2]

$$\begin{aligned} U(\omega) &= D(\omega)\langle \varepsilon(\omega) \rangle \\ &= \frac{kT}{2\pi v} \end{aligned} \qquad (4)$$

The power per unit frequency is then simply:[3]

$$\begin{aligned} P(\omega) &= vU(\omega) \\ &= \frac{kT}{2\pi} \end{aligned} \qquad (5)$$

OR

$$P(\nu) = kT \qquad (6)$$

This is the power per unit frequency *absorbed* by the resistor. By the *principle of detailed balance* this must be equal to the power per unit frequency *emitted* by the resistor. The thermal electromotive force generated by the resistor sets up a current $I = V/2R$ in the transmission line. Thus, the power absorbed by the resistor at the other end is

$$P(\nu) = \langle I^2(\nu) \rangle R \qquad (7a)$$

$$= \left\langle \frac{V^2(\nu)}{4R^2} \right\rangle R \qquad (7b)$$

$$= \frac{\langle V^2(\nu) \rangle}{4R} \qquad (7c)$$

Equating Eq. 6 to Eq. 7c and then solving for the mean-square voltage per unit frequency gives:

$$\langle V^2(\nu) \rangle = 4RkT \qquad (8)$$

By integrating the expression above over the accesible frequency range, we arrive at the Nyquist Theorem:

$$\boxed{\langle V^2 \rangle = 4kTR\Delta\nu \qquad (9)}$$

[1]The Hamiltonian (per unit volume) for an electromagnetic wave is given by $\mathcal{H} = \frac{1}{8\pi}(\mathbf{E^2} + \mathbf{B^2})$.

[2]$U(\omega)$ is a one-dimensional energy density.

[3]Recall that the energy density is equivalent to a force.

B. Microscopic Derivation

Consider a conductor of resistance R with a charge carrier density N having a relaxation time τ_c. The conductor has length ℓ and cross-sectional area A. The voltage V across the conductor is

$$\begin{aligned} V &= IR && (10a)\\ &= RAj && (10b)\\ &= RANe\langle u\rangle && (10c)\end{aligned}$$

where I is the current, j is the current density, e is the charge on an electron, and $\langle u\rangle$ is the drift speed along the conductor.

Noting that $NA\ell$ is the total number of electrons in the conductor,

$$\sum_i u_i = NA\ell\langle u\rangle \tag{11}$$

Solving for $\langle u\rangle$ in Eq. 11 and substituting the resulting expression into Eq. 10c gives,

$$V = \sum_i V_i = \frac{Re}{\ell}\sum_i u_i \tag{12}$$

where u_i and V_i are random variables.

The spectral density $J(\nu)$ has the property that in the frequency interval $\Delta\nu$

$$\langle V_i^2\rangle = J(\nu)\Delta\nu \tag{13}$$

The correlation function can be written as

$$\begin{aligned} C(\tau) &= \langle V_i(t)V_i(t+\tau)\rangle && (14a)\\ &= \langle V_i^2(t)\rangle \exp(-\tau/\tau_c) && (14b)\end{aligned}$$

where τ is an arbitrary time interval.

By substiting Eq. 14b and Eq. 12 into the Wiener-Khintchine theorem Eq. 15a, the spectral density is

$$\begin{aligned} J(\nu) &= 4\int_0^\infty C(\tau)\cos(2\pi\nu\tau)\,d\tau && (15a)\\ &= 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\int_0^\infty exp(-\tau/\tau_c)\cos(2\pi\nu\tau)\,d\tau && (15b)\\ &= 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\frac{\tau_c}{1+(2\pi\nu\tau_c)^2} && (15c)\\ &\approx 4\left(\frac{Re}{\ell}\right)^2\langle u^2\rangle\tau_c && (15d)\\ &\approx 4\left(\frac{Re}{\ell}\right)^2\left(\frac{kT}{m}\right)\tau_c && (15e)\end{aligned}$$

where $\langle u^2\rangle = kT/m$ by the equipartition theorem. Note that for metals at room temperature $\tau_c < 10^{-13}$, thus from the DC through the microwave range $2\pi\nu\tau_c \ll 1$.

Thus the mean-square voltage in the frequency range $\Delta\nu$ equals:

$$\begin{aligned} \langle V^2\rangle &= NA\ell\langle V_i^2\rangle && (16a)\\ &= NA\ell J(\nu)\Delta\nu \quad \text{using Eq. 13} && (16b)\\ &= NA\ell 4\left(\frac{Re}{\ell}\right)^2\left(\frac{kT}{m}\right)\tau_c\Delta\nu \quad \text{using Eq. 15e} && (16c)\\ &= 4\left(\frac{Ne^2\tau_c}{m}\right)\frac{A}{\ell}R^2kT\Delta\nu && (16d)\end{aligned}$$

Using a result from conductivity theory $\sigma = Ne^2\tau_c/m$ and the elementary relation $R = \frac{\ell}{\sigma A}$ [5]:

$$\langle V^2\rangle = 4\underbrace{\sigma\frac{A}{\ell}}_{1/R}R^2kT\Delta\nu \tag{17}$$

We have once again arrived at the Nyquist Theorem:

$$\boxed{\langle V^2\rangle = 4kTR\Delta\nu \qquad (18)}$$

The Nyquist Theorem is a special case of the general connection existing between fluctuations (random variables) and dissipation in physical systems. Brownian motion lends itself to a similar analysis [6], [7].

III. EXPERIMENTS

This section describes the experimental apparatus used, the calibration performed and the measurements that were recorded.

A. Apparatus

Figure 5 is a diagram of the experimental apparatus used to measure the Johnson Noise.[4] An inverted beaker shielded the resistor R which was mounted on the terminal of the aluminum box. The resistor is connected to the measurement chain through two switches (SW1 and SW2). A Hewlett-Packard HP54601A digital oscilloscope was used to measure the root-mean-square (RMS) voltage generated by the resistor. Because the Johnson Noise signals are in the microvolt range, a low-noise amplifier (PAR 113) was used to produce millivolt signals detectable by the digital oscilloscope. A band-pass filter (Krohn-Hite 3202R) was used to prevent thermal noise outside the frequency range 1 KHz – 50 KHz from being amplified.[5] A Tektronix Function Generator (FG) 504

[4] Figure 5 was scanned-in from the junior lab guide [8].

[5] Signals outside this frequency range could not be properly amplified by the PAR 113.

provided sinusoidal calibration signals. The FG and the Kay attenuator were used to calibrate the measurement chain.

Several steps were taken to filter out extraneous noise from the experimental apparatus. At all times the digital oscilloscope was kept at least five feet from the noise source, otherwise the variable magnetic field from its beam-control coil would produce undesirable electrical oscillations in our noise measurements. Coaxial cables were also kept as short as possible to keep minimize electrical interference.

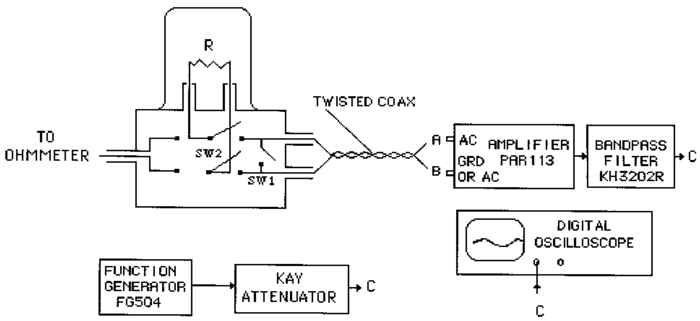

FIG. 5. Experimental apparatus.

B. Calibration of Measurement Chain

1. Test signal RMS voltage

The amplitude of the sinusoidal signal produced from the FG was adjusted so that the RMS voltage V_{RMS} as measured on the digital oscilloscope was approximately 2 volts. The RMS voltage of the FG sinusoidal signal was recorded over the range passed by the Krohn-Hite Filter (refer to Figure 6). It was confirmed that the RMS voltage varied slightly over the frequency range of interest.

2. Gain of measurement chain

The sinusoidal test signal was fed through the Kay attenuator set to 60 dB (1000) of attenuation to the 'A' input of the PAR amplifier (set to 1K) with 'B' input grounded. The RMS voltages out of the Krohn-Hite filter were measured over a 100 kHz frequency range. The gain squared $[g(\nu)]^2$ was small at very low frequency, then drastically increased to unity around 5 kHz (refer to Figure 7). As expected at higher frequency (> 50 kHz) the gain squared roll off considerably.

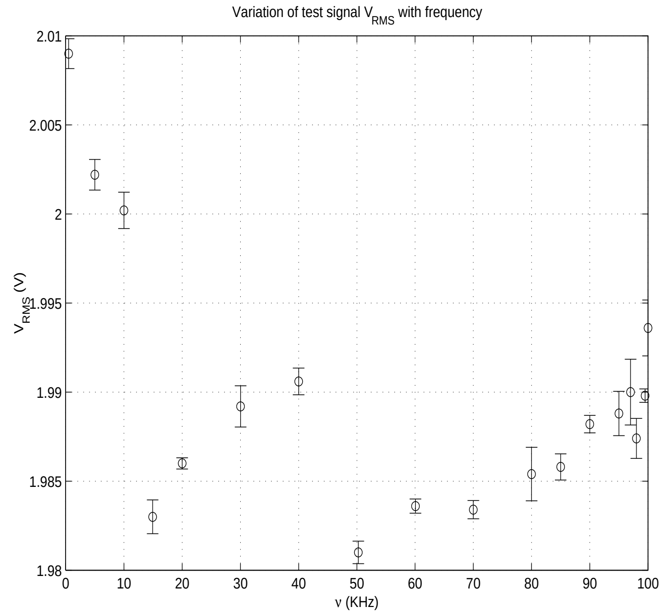

FIG. 6. RMS voltage V_{RMS} produced by function generator as a function of frequency.

C. Resistance Dependence of Johnson Noise

With the PAR amplifier set to 1K, typical RMS voltages out of the Krohn-Hite filter were in the millivolt range. The component of the noise V_S not generated by the resistor but by the amplifier itself was measured by:

1. Opening SW2.
2. Unplugging the connections to the ohmmeter and temperature meter.
3. Shorting the resistor with SW1.

The total RMS voltage V_R was measured with the shorting switch SW1 open. Because all the contributions to the measure RMS voltage are statistically uncorrelated, *they add in quadrature*. Thus, mean square Johnson noise of the resistor is given by,

$$V_{\mathrm{Jo}}'^2 = V_R^2 - V_S^2 \tag{19}$$

where V_R and V_S are the RMS voltages measured with the SW1 open and closed, respectively. The resistance R was measured using a digital multimeter after each noise measurement.

D. Temperature Dependence of Johnson Noise

The Johnson noise of a 22.2 kΩ resistor was measured at liquid N_2 temperature −160°C to 150°C. High temperatures were obtained by mounting the inverted aluminum box and placing it on a cylindrical oven. The temperature was adjusted by using a Variac. Low temperatures

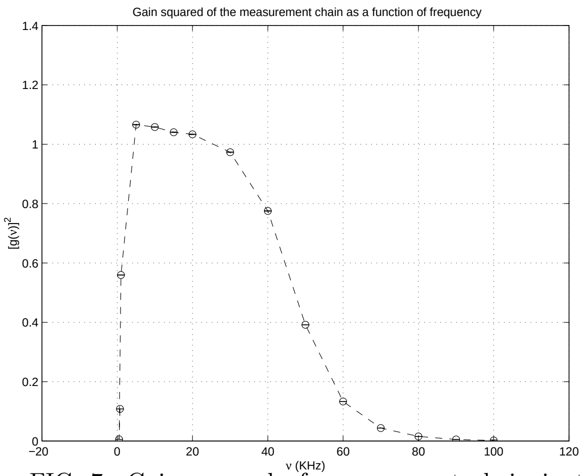

FIG. 7. Gain squared of measurement chain in the frequency range (0.5 KHz – 100 kHz.) **NOTE:** The dotted line is not a fitted function. Its purpose is tom emphasize a trend in the gain squared. The gain squared has been normalized such that the value of $[g(\nu)]^2 = 1$ corresponds to a gain of 1000.

were obtained by inverting the aluminum box and placing it on a liquid N_2 filled dewar flask. The temperature was varied in an *ad-hoc* manner by raising and lowering the aluminum box into the dewar flask as needed.

IV. RESULTS AND DISCUSSION

The first subsection explicitly connects the Nyquist Theorem with the experimental setup at hand. The last two subsections describe the results of the subsections III C & III D, respectively.[6]

A. Derivation of RMS thermal voltage at the terminal of an RC circuit

The resistor and coaxial cables that are connected to the PAR amplifier can be modeled as the circuit depicted in Figure 8. The equivalent circuit is composed of a fluctuating thermal electromotive force V_{Jo} with an ideal resistor R and a capacitor C in a simple lowpass filter configuration.

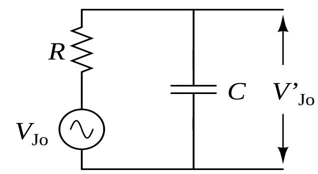

FIG. 8. Equivalent circuit of the electromotive force across a conductor of resistance R connected to the measuring device with cables having capacitance C.

In sinusoidal steady state, impedances can be used to treat the circuit as a voltage divider.

$$V'_{\text{Jo}} = \frac{(i\omega C)^{-1}}{(i\omega C)^{-1} + R} g(\omega) V_{\text{Jo}} \tag{20a}$$

$$= \frac{1}{1 + i\omega C} g(\omega) V_{\text{Jo}} \tag{20b}$$

The RMS thermal voltage is the magnitude of Eq. 20b:

$$V'^2_{\text{Jo}} = \frac{[g(\nu)]^2 V^2_{\text{Jo}}}{1 + (2\pi\nu RC)^2} \tag{21}$$

The Johnson Noise is equation Eq. 21 summed over the accessible frequencies,

$$V'^2_{\text{Jo}} = V^2_{\text{Jo}} \underbrace{\int_0^\infty \frac{[g(\nu)]^2}{1 + (2\pi\nu RC)^2}\, d\nu}_{G} \tag{22}$$

In this experiment, the integral in Eq. 22 was numerical evaluated using the data collected in the calibration of the measurement chain (Figure 7). The capacitance C was approximated at 60 pF from considerations of the amount of coaxial cable used and its known capacitance per unit length, 30.8 pF/feet. The Nyquist Theorem expressed in terms of the present variables is arrived at by taking Eq. 9(or 18) and making the substitutions: $\langle V^2 \rangle \to V'^2_{\text{Jo}}$ and $\Delta\nu \to G$.

$$V'^2_{\text{Jo}} = 4kTRG \tag{23}$$

B. Determination of the Boltzmann Constant

The RMS voltage was measured for eight metal film resistors (whose values ranged from 20 kΩ to 10^3 kΩ) at room temperature. Figure 9 is a plot of V'^2_{Jo} against R. The Boltzmann constant was calculated by solving for k in Eq. 23. The experimentally determined value of the Boltzmann constant, $1.37 \pm 0.06 \times 10^{-23}$ J/K, is in good agreement with the accepted value 1.38×10^{-23} J/K.

[6]Note that Boltzmann constant is calculated in the last *two* subsections.

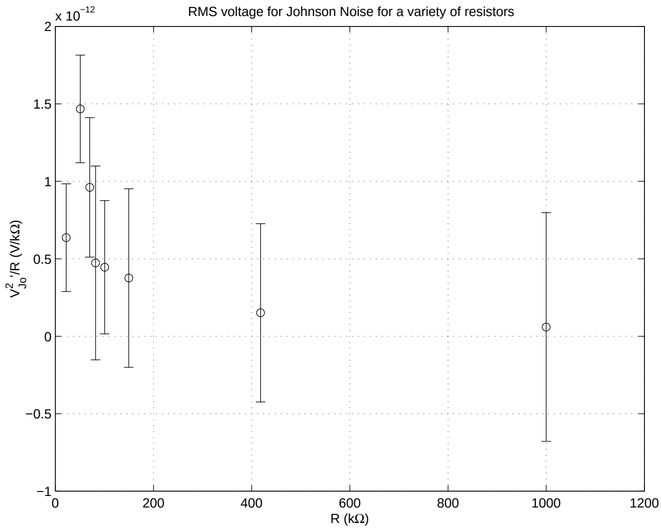

FIG. 9. Resistance dependence of Johnson Noise V'_{Jo}.

C. Determination of the Absolute Zero on Centigrade Scale

The RMS voltage for 22.2 kΩ resistor was measured at fourteen temperatures ranging from $\sim -160°$C to $\sim 150°$C at approximate intervals of 25°C Figure 10 is a least-squares fit of $V'^2_{\rm Jo}/4RG$ vs. T. The slope of the line gives the Boltzmann constant and the T-intercept is the centigrade temperature of absolute zero. The Boltzmann constant was determined to be $1.363 \pm 0.025 \times 10^{-23}$ J/K and centigrade temperature of absolute zero was extrapolated to $-265.5 \pm 6.9°$C. Both experimentally determined values are in good agreement with their accepted values of 1.38×10^{-23} J/K and -273.15K, respectively.

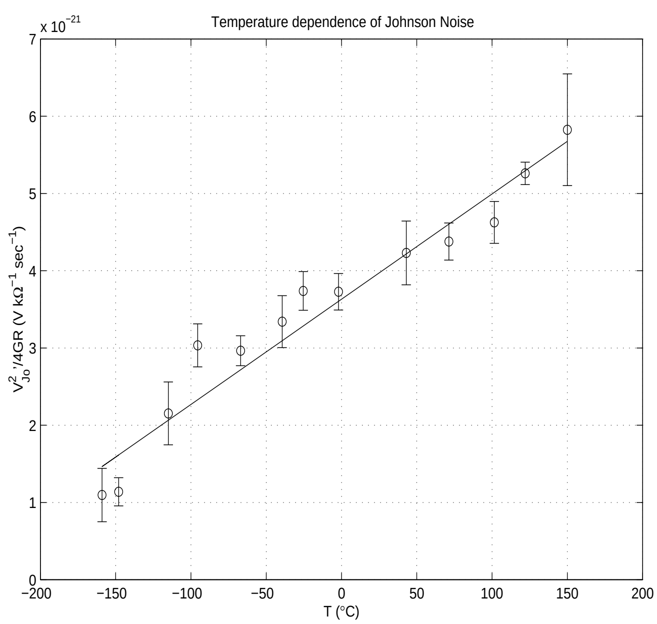

FIG. 10. Temperature dependence of Johnson Noise V'_{Jo}.

V. CONCLUSIONS

Johnson Noise belongs to a broader category of stochastic phenomena which have been of research interest for decades. Measurement of the thermal noise in resistors provided a means to calculate the Boltzmann constant and the centigrade temperature of absolute zero. Because there are inherent difficulties in measuring thermal noise, the Boltzmann constant was measured to an accuracy of ~ 4 %.[7] Alternate methods of implementing a undergraduate physics experiment on Johnson Noise are described in the literature (e.g. [9]).

ACKNOWLEDGMENTS

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[7] In his original paper, Dr. J. B. Johnson measured the Boltzmann constant within 8 % of the accepted value.